

# On Some $q$ -Analogues of the Natural Transform and Further Investigations

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## Abstract

Some  $q$ -analogues of classical integral transforms have recently been investigated by many authors in diverse citations. The  $q$ -analogues of the Natural transform are not known nor used. In the present paper, we are concerned with definitions and investigations of the  $q$ -theory of the Natural transform and some applications. We present two types of  $q$ -analogues of the cited transform on given sets and get results of the nominated analogues for certain class of functions of special type. We declare here that given results are new and they complement recent known results related to  $q$ -Laplace and  $q$ -Sumudu transforms. Over and above, we present some supporting examples to illustrate effectiveness of the given results .

**Keywords :**  $q$ -Sumudu transform;  $q$ -Laplace transform;  $q$ -Natural transform;  $q$ -Bessel function.

## 1 Introduction

The study of  $q$ -analogues of classical integral transforms has not yet been developed to a great extent. This partially can be explained by the fact that one is not very familiar with the  $q$ -theory and that basic  $q$ -integral transforms do not occur frequently in physics. It by no means our aim to give in this paper a new general results in the theory of  $q$ -calculus. But we restrict our selves to the necessary theory to give some  $q$ -analogues of an integral transform named as the Natural transform and to estimate the transform relevant properties. The classical theory of the Natural transform is closely related to the classical theory of Laplace and Sumudu transforms, two of the best known of all integral transforms. The Natural transform for functions of exponential order is defined over the set  $A$ ,

$$A = \left\{ f(t) \left| \exists M, \tau_1 \text{ and/or } \tau > 0: |f(t)| < M e^{\frac{|t|}{\tau}}, t \in (-1)^j \times [0, \infty), j = 1, 2 \right. \right\} \quad (1)$$

by the integral equation

$$N(f)(u; v) = \int_0^\infty f(ut) \exp(-vt) dt, \quad (2)$$

where  $\operatorname{Re} v > 0$  and  $u \in (-\tau_1, \tau_2)$ ,  $u$  and  $v$  being the transform variables.

The Natural transform strictly converges to the Sumudu transform (See [28, 29, 30] ) for  $v = 1$  and it strictly converges to the Laplace transform for  $u = 1$ . Further fundamental properties of this transform and its application to differential equations are given by [14] and [15, 16, 17], respectively.

We organize this paper as follows. In section 2, we recall some definitions and notations from the  $q$ -calculus. In Section 3, we specify the  $q$ -analogues of the Natural transform in terms of a series representation. In Section 4, we apply the first of two analogues of

the Natural transform to a certain class of special functions and pick up some results by considering  $q$ -Laplace and  $q$ -Sumudu transforms. In Section 5, we figure out some values of the second representation of the  $q$ -transform and extend the resulting theorems to the case of  $q$ -Laplace and  $q$ -Sumudu transforms. In Section 6 and 7, we derive certain results concerning Fox's  $H_q$ -function and propose some counter examples as an application to the previous theory.

## 2 The $q$ -Calculus

For the convenience of the reader, we provide a summary of mathematical notations used in this paper. Throughout this paper, wheresoever it appears,  $q$  satisfies the condition that  $0 < q < 1$ .

The  $q$ -calculus begins with the definition of the  $q$ -analogue  $d_q f(x)$  of the differential of the function, i.e.,

$$d_q f(x) = f(x) - f(qx),$$

and the  $q$ -analogue of the derivative,

$$\frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{(q-1)x}.$$

On certain additional requirements, the  $q$ -analogue may be unique, but sometimes it is useful to consider several  $q$ -analogues of the same object.

The  $q$ -analogues of an integer  $n$  ( $q$ -integer), a factorial of  $n$  ( $q$ -factorial of  $n$ ), and the binomial coefficient  $\binom{n}{k}$  ( $q$ -binomial coefficient) are respectively given as

$$[n]_q = \frac{1-q^n}{1-q}, \quad ([n]_q)! = \begin{cases} \prod_{k=1}^n [k]_q, & n = 1, 2, 3, \dots \\ 1, & n = 0 \end{cases} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{k=1}^n \frac{1-q^{n-k+1}}{1-q^k}.$$

Clearly,

$$\lim_{q \rightarrow 1} [n]_q = n, \quad \lim_{q \rightarrow 1} ([n]_q)! = n! \quad \text{and} \quad \lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

If  $\alpha \in \mathbb{C}$ , then the  $q$ -analogue of  $\alpha$  is given as  $\frac{1-q^\alpha}{1-q}$  and, it sometimes makes sense when

$$\alpha \text{ is not, } [\infty]_q = \frac{1}{1-q}.$$

If  $n \in \mathbb{N}$ , then the  $q$ -analogue of  $(x+a)^n$  and the derivative are respectively given as

$$(x+a)_q^n = \prod_{j=0}^{n-1} (x+q^j a) \quad \text{and} \quad D_q(x+a)_q^n = [n]_q (x+a)_q^{n-1}, \quad (x+a)_q^0 = 1 \quad \Bigg\}. \quad (3)$$

If  $x = 1$  and  $a = x$ , then the above formula makes sense for  $n = \infty$ , giving

$$(1+x)_q^\infty = \prod_{k=0}^{\infty} (1+q^k x). \quad (4)$$

The  $q$ -Jackson integral from 0 to  $a$  is given by Jackson [11] as

$$\int_0^a f(x) d_q x = (1-q)a \sum_{k=0}^{\infty} f(aq^k) q^k, \quad (5)$$

provided the sum converges absolutely.

The  $q$ -Jackson integral in a generic interval  $[a, b]$  is given by [11]

$$\int_b^a f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \quad (6)$$

The improper integral is defined as [5]

$$\int_0^{\frac{\infty}{A}} f(x) d_q x = (1-q) \sum_{n \in \mathbb{Z}} \frac{q^n}{A} f\left(\frac{q^n}{A}\right). \quad (7)$$

The  $q$ -analogues of the gamma function are defined by [5]

$$\begin{aligned} \Gamma_q(\alpha) &= \int_0^{\frac{1}{1-q}} x^{\alpha-1} E_q(q(1-q)x) d_q x, \\ {}_q\Gamma(\alpha) &= K(A; \alpha) \int_0^{\frac{\infty}{A(1-q)}} x^{\alpha-1} e_q(-(1-q)x) d_q x, \end{aligned}$$

where  $\alpha > 0$  and that

$$K(A; t) = A^{t-1} \frac{(-q/A; q)_\infty}{(-q^t/A; q)_\infty} \frac{(-A; q)_\infty}{(-Aq^{1-t}; q)_\infty}, \quad (8)$$

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n. \quad (9)$$

for all  $t \in \mathbb{R}$ .

The useful notations we need here are [10]

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x} \quad \text{and} \quad (a; q)_t = \frac{(a; q)_\infty}{(aq^t; q)_\infty}, \quad t \in \mathbb{R}. \quad (10)$$

The  $q$ -analogues of the exponential function are given as

$$\left. \begin{aligned} E_q(t) &= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} t^n = (t; q)_\infty, \quad t \in \mathbb{C} \\ e_q(t) &= \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} t^n = \frac{1}{(t, q)_\infty}, \quad t < 1. \end{aligned} \right\} \quad (12)$$

The  $q$ -analogues of the hypergeometric function are defined in two ways as

$${}_r\phi_s \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n} \frac{z^n}{(q; q)_n} \quad (13)$$

and

$${}_{m-k}\Phi_{m-1} \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k} \\ b_1, b_2, \dots, b_{m-1} \end{matrix} \middle| q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{m-k}; q)_n}{(b_1, \dots, b_{m-1}; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^k \frac{z^n}{(q; q)_n}, \quad (14)$$

where  $(a_1, a_2, \dots, a_p; q)_n = \prod_{k=0}^p (a_k, q)_n$ .

For  $|z| < 2$ , some  $q$ -analogues of the Bessel function are defined as

$$J_v^{(1)}(z; q) = \frac{(q^{v+1}; q)_\infty}{(q; q)_\infty} \left(\frac{z}{2}\right)^2 {}_2\Phi_1 \left[ \begin{matrix} 0 & 0 \\ q^{v+1} \end{matrix} \middle| q, \frac{-z^2}{2} \right] = \left(\frac{z}{2}\right)^v \sum_{n=0}^{\infty} \frac{\left(\frac{-z^2}{4}\right)^n}{(q; q)_{v+n}} (q; q)_n, \quad (15)$$

$$J_v^{(2)}(z; q) = \frac{(q^{v+1}; q)_\infty}{(q; q)_\infty} \left(\frac{z}{2}\right)^v {}_0\Phi_1 \left[ \begin{matrix} - \\ q^{v+1} \end{matrix} \middle| q, \frac{-q^{v+1}z^2}{4} \right] = \left(\frac{z}{2}\right)^v \sum_{n=0}^{\infty} \frac{q^{n(n+v)}}{(q; q)_{v+n} (q; q)_n} \left(\frac{-z^2}{4}\right)^n, \quad (16)$$

$$J_v^{(3)}(z; q) = \frac{(q^{v+1}; q)_\infty}{(q; q)_\infty} z^v {}_1\Phi_1 \left[ \begin{matrix} 0 \\ q^{v+1} \end{matrix} \middle| q, qz^2 \right] = z^v \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(q; q)_{v+n} (q; q)_n}. \quad (17)$$

### 3 The $q$ -Analogues of the Natural Transform

Theory and applications of  $q$ -integral transforms are evolving rapidly over the recent years. Since Jackson [11] presented a precise definition of so-called  $q$ -Jackson integral and developed  $q$ -calculus in a systematic way. It was well known that, in the literature, there are two types of  $q$ -analogues of integral transforms studied in detail by many authors in the recent past such as Abdi [2], Hahn [18], Purohit and Kalla [3], Albayrak [25], Uçar and Albayrak [4], Albayrak et al. [5] and [6], Yadav and Purohit [7], Fitouhi and Bettaibi [8] and [9] and many others, to mention but a few.

In this section of this paper we deem it proper to give the definition of the  $q$ -analogues of the Natural transform as in the following definition.

DEFINITION 1. Let  $\hat{A}$  and  $\check{A}$  be defined by

$$\hat{A} = \left\{ f(t) \middle| \exists M, \tau_1 \text{ and/or } \tau > 0: |f(t)| < M E_q \left( \frac{|t|}{\tau_j} \right), t \in (-1)^j \times [0, \infty), j = 1, 2 \right\}$$

and

$$\check{A} = \left\{ f(t) \middle| \exists M, \tau_1 \text{ and/or } \tau > 0: |f(t)| < M e_q \left( \frac{|t|}{\tau_j} \right), t \in (-1)^j \times [0, \infty), j = 1, 2 \right\},$$

respectively. Then, we have the following definitions.

(i) Over the set  $\hat{A}$ , we define the  $q$ -analogue of the Natural transform of first type as

$$N_q(f)(u; v) = \frac{1}{(1-q)u} \int_0^{\frac{u}{v}} f(t) E_q \left( q \frac{v}{u} t \right) d_q t, \quad (18)$$

provided the integral exists.

(ii) Over the set  $\check{A}$ , we define the  $q$ -analogue of the Natural transform of type two as

$${}_q N(f)(u; v) = \frac{1}{(1-q)} \int_0^\infty f(t) e_q \left( -\frac{v}{u} t \right) d_q t, \quad (19)$$

when the integral exists.

It seems very beneficial to us to notice the following relations

$$\begin{aligned} N_q(f)(1; v) &= (L_q f)(v), & {}_q N(f)(1; v) &= ({}_q L f)(v), \\ N_q(f)(u; 1) &= (S_q f)(u), & {}_q N(f)(u; 1) &= ({}_q S f)(u), \end{aligned} \quad (20)$$

where  $L_q(S_q)$  and  ${}_q L({}_q S)$  are respectively the  $q$ -analogues of the Laplace (Sumudu) transforms of first (second) types; see, for example, [4] (resp., [6]).

In terms of Jackson integral series representation, the  $q$ -analogue of (18) can be expressed as

$$N_q(f)(u; v) = \frac{(q; q)_\infty}{v} \sum_{k \geq 0} \frac{q^k f\left(\frac{u}{v} q^k\right)}{(q; q)_k}, \quad (21)$$

whereas, the  $q$ -analogue of (19) can similarly be performed in terms of that series as

$${}_q N(f)(u; v) = \sum_{k \in \mathbb{Z}} \frac{(q; q)_\infty}{v} \frac{f(q^k)}{\left(-\frac{u}{v} q^k; q\right)_\infty}.$$

Hence, on parity of the fact  $(a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}$  ( for  $a = \frac{-v}{u}$ ), the previous equation has the series representation

$${}_q N(f)(u; v) = \frac{1}{\left(-\frac{v}{u}; q\right)_\infty} \sum_{k \in \mathbb{Z}} q^k f(q^k) \left(-\frac{u}{v}; q\right)_k \quad (22)$$

that we shall use in later investigations.

## 4 $N_q$ of Some Special Functions

In this section of this paper, we apply the analogue  $N_q$  for certain functions of special type and extend the work to  $q$ -Laplace and  $q$ -Sumudu transforms. We assume the functions, unless otherwise stated, are of power series form,

$$f(x) = \sum_{n \geq 0} A_n x^n, \quad (23)$$

where  $A_n$  is some bounded sequence .

In what follows, we establish the following three main theorems of this section.

**THEOREM 1.** *Let  $\alpha$  be a positive real number and  $f(x) = \sum_{n \geq 0} A_n x^n$ . Then, we have*

$$N_q(x^{\alpha-1} f(x))(u; v) = \frac{(1-q)^{\alpha-1}}{v^\alpha} u^{\alpha-1} \sum_{n \geq 0} A_n \frac{u^n}{v^n} (1-q)^n \Gamma_q(\alpha + n). \quad (24)$$

**PROOF** *Let  $\alpha$  be a positive real number and  $f(x) = \sum_{n \geq 0} A_n x^n$  be given. Then, on aid of (21), we write*

$$\begin{aligned} N_q(x^{\alpha-1} f(x))(u; v) &= \frac{(q; q)_\infty}{v} \sum_{k \geq 0} q^k \frac{\left(\frac{u}{v} q^k\right)^{\alpha-1} f\left(\frac{u}{v} q^k\right)}{(q; q)_k} \\ &= \frac{u^{\alpha-1}}{v^\alpha} (q; q)_\infty \sum_{k \geq 0} \frac{q^{\alpha k}}{(q; q)_k} \sum_{n \geq 0} A_n \left(\frac{u}{v} q^k\right)^n \\ &= \frac{u^{\alpha-1}}{v^\alpha} (q; q)_\infty \sum_{n \geq 0} A_n \left(\frac{u}{v}\right)^n \sum_{k \geq 0} \frac{q^{k(\alpha+n)}}{(q; q)_k}. \end{aligned} \quad (25)$$

Hence, taking into account (11), (25) simply reveals

$$\begin{aligned} N_q \left( x^{\alpha-1} f(x) \right) (u; v) &= \frac{u^{\alpha-1}}{v^\alpha} (q; q)_\infty \sum_{n \geq 0} A_n \left( \frac{u}{v} \right)^n e_q \left( q^{\alpha+n} \right) \\ &= \frac{u^{\alpha-1}}{v^\alpha} (q; q)_\infty \sum_{n \geq 0} A_n \frac{u^n}{v^n} \frac{1}{(q^{(\alpha+n)}; q)_\infty}. \end{aligned} \quad (26)$$

Therefore, by aid of the Equations (10) and (26) we get that

$$\begin{aligned} N_q \left( x^{\alpha-1} f(x) \right) (u; v) &= \frac{u^{\alpha-1}}{v^\alpha} (q; q)_\infty \sum_{n \geq 0} A_n \left( \frac{u}{v} \right)^n e_q \left( q^{\alpha+n} \right) \\ &= \frac{(1-q)^{\alpha-1}}{v^\alpha} u^{\alpha-1} \sum_{n \geq 0} A_n \frac{u^n}{v^n} \frac{\Gamma_q(\alpha+n)}{(1-q)^{-n}} \\ &= \frac{(1-q)^{\alpha-1}}{v^\alpha} u^{\alpha-1} \sum_{n \geq 0} A_n \frac{u^n (1-q)^n}{v^n} \Gamma_q(\alpha+n). \end{aligned}$$

This completes the proof of the theorems.

**THEOREM 2.** *Let  $\alpha$  be a positive real number. Then, the following hold.*

(i) *Let  $\Gamma_q(\alpha)$  be the  $q$ -gamma function of the first type. Then, we have*

$$N_q \left( x^{\alpha-1} f(x) \right) (u; v) = \frac{(1-q)^{\alpha-1}}{v^\alpha} u^{\alpha-1} \Gamma_q(\alpha).$$

(ii) *Let  $a \in \mathbb{R}$  and  $f(x) = {}_{m-k}\Phi_{m-1} \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k} \\ b_1, b_2, \dots, b_{m-1} \end{matrix} \middle| q, ax \right]$ . Then, we have*

$$\begin{aligned} (N_q x^{\alpha-1} f(x)) (u; v) &= \frac{\Gamma_q(\alpha) (1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} \\ &\quad {}_{m-k+1}\Phi_m \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k} q^\alpha \\ b_1, b_2, \dots, b_{m-1}, 0 \end{matrix} \middle| q, \frac{au}{v} \right]. \end{aligned} \quad (27)$$

**PROOF OF (i)** By assuming  $A_0 = 1$  and  $A_n = 0, \forall n \geq 1$ , it follows from (23) that  $f(x) = 1$ .

Hence, the first part obviously follows.

**PROOF OF (ii)** Appealing to the  $q$ -analogue (14),  $f(x)$  can fairly be written as

$$f(x) = \sum_{n \geq 0} \frac{(a_1, a_2, \dots, a_{m-k}; q)_n}{(b_1, b_2, \dots, b_{m-1}; q)_n} \left( (-1)^n q^{\frac{n(n-1)}{2}} \right) \frac{q^n}{(q; q)_n} x^n.$$

On setting

$$A_n = \frac{(a_1, a_2, \dots, a_{m-k}; q)_n}{(b_1, b_2, \dots, b_{m-1}; q)_n} \left( (-1)^n q^{\frac{n(n-1)}{2}} \right)^k a^n, \quad (28)$$

we get the power series representation of type (23). Hence, Theorem 1 reveals

$$N_q \left( x^{\alpha-1} f(x) \right) (u; v) = \frac{(1-q)^{\alpha-1}}{v^\alpha} u^{\alpha-1} \sum_{n \geq 0} A_n \frac{u^n}{v^n} (1-q)^n \Gamma_q(\alpha+n).$$

Therefore, on using the identity of the gamma function [1]

$$\Gamma_q(x+j) = \frac{(q^x; q)_j}{(1-q)_j} \Gamma_q(x), \quad (29)$$

the previous equation consequently gives

$$\begin{aligned}
 N_q \left( x^{\alpha-1} f(x) \right) (u; v) &= \frac{(1-q)^{\alpha-1}}{v^\alpha} u^{\alpha-1} \sum_{n \geq 0} A_n \frac{u^n}{v^n} (q^\alpha; q)_n \Gamma_q(\alpha) \\
 &= \frac{\Gamma_q(\alpha) (1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} \sum_{n \geq 0} A_n \frac{u^n}{v^n} (q^\alpha; q)_n \frac{u^n}{v^n} \\
 &= \frac{\Gamma_q(\alpha) (1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} {}_{m-k+1}\Phi_m \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k} q^\alpha \\ b_1, b_2, \dots, b_{m-1}, 0 \end{matrix} \middle| q, \frac{au}{v} \right].
 \end{aligned}$$

This completes the proof of the theorem.

An inspection of the previous two main theorems leads to the following list of inclusions:

On setting  $k = m = 1$ , Theorem 2 (ii) (for  $\alpha = 1$ ) yields

$$N_q (E_q(ax)) (u; v) = \frac{1}{v} {}_1\Phi_1 \left[ \begin{matrix} q \\ 0 \end{matrix} \middle| q, \frac{au}{v} \right]. \quad (30)$$

Similarly, by inserting  $\alpha = 1$ , Theorem 2 (i) instantly shows

$$(N_q(1)) (u; v) = \frac{1}{v}. \quad (31)$$

Moreover, by setting  $\alpha = n + 1$  in the first part of the theorem, it yields

$$N_q(x^n) (u; v) = \frac{(1-q)^n}{v^{n+1}} u^n ([n]_q)! . \quad (32)$$

Therefore, (31) and (32) when designated reveal

$$N_q(1 + x^2 + \dots + x^n) (u; v) = \sum_{k \geq 0} \frac{(1-q)^k}{v^{k+1}} u^k ([k]_q)! .$$

We finally establish the main third theorem of this section.

**THEOREM 3.** *Let  $f(x)$  be given as  $f(x) = {}_r\phi_p \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} \middle| q, ax \right]$  and  $\alpha > 0$ . Then, we have*

$$N_q(x^{\alpha-1} f(x)) (u; v) = \frac{\Gamma_q(\alpha) (1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} {}_{r+1}\phi_p \left[ \begin{matrix} a_1, a_2, \dots, a_r, q^\alpha \\ b_1, b_2, \dots, b_p \end{matrix} \middle| q, a \frac{u}{v} \right]. \quad (33)$$

**PROOF** By taking into account (13),  $f(x)$  is given the series representation

$$f(x) = \sum_{n \geq 0} \frac{(a_1, a_2, \dots, a_{m-k}; q)_n}{(b_1, b_2, \dots, b_{m-1}, q)_n} \frac{a^n}{(q; q)_n} x^n.$$

On setting  $A_n = \frac{(a_1, a_2, \dots, a_{m-k}; q)_n}{(b_1, b_2, \dots, b_{m-1}, q)_n} \frac{a^n}{(q; q)_n}$  and using (21) and (29), we get

$$\begin{aligned}
 N_q(x^{\alpha-1} f(x)) (u; v) &= \frac{(1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} \sum_{n \geq 0} A_n \frac{u^n}{v^n} (1-q)^n \Gamma_q(\alpha + n) \\
 &= \frac{\Gamma_q(\alpha) (1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} \sum_{n \geq 0} A_n (q^\alpha; q)_n \frac{u^n}{v^n} \\
 &= \frac{\Gamma_q(\alpha) (1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} {}_{r+1}\phi_p \left[ \begin{matrix} a_1, a_2, \dots, a_r, q^\alpha \\ b_1, b_2, \dots, b_p \end{matrix} \middle| q, a \frac{u}{v} \right].
 \end{aligned}$$

This completes the proof of the theorem.

By setting  $p = 0$ ,  $\alpha = 1$  and  $r = 0$  in (33), the above theorem leads to

$$N_q(e_q(ax))(u; v) = \frac{1}{v} {}_1\phi_0\left[q; - \left|q, a\frac{u}{v}\right|\right] = \frac{1}{v} \sum_{n \geq 0} \left(\frac{au}{v}\right)^n = \frac{1}{v - au}, \quad |au| < v. \quad (34)$$

Further, by fixing  $v = 1$  and taking account of (20), (34) spreads the result to the case of  $q$ -Sumudu transform giving

$$S_q(e_q(ax))(u) = \frac{1}{1 - au}, \quad |au| < 1.$$

Similarly, by fixing  $u = 1$  and consulting (20) yield the following case of  $q$ -Laplace transform

$$L_q(e_q(ax))(v) = L_q(e_q(ax))(v) = \frac{1}{v - a}, \quad |a| < v.$$

From above investigations, we, further, deduce

$$N_q(\sin_q ax)(u; v) = N_q\left(\frac{e_q(iax) - e_q(-iax)}{2i}\right)(u; v) = \frac{au}{v^2 + a^2u^2}, \quad |au| < v, \quad (35)$$

and

$$N_q(\cos_q ax)(u; v) = N_q\left(\frac{e_q(iax) + e_q(-iax)}{2i}\right)(u; v) = \frac{v}{v^2 + a^2u^2}, \quad |au| < v. \quad (36)$$

Hence, by virtue of (35) and (36) we state without proof the following corollary.

**COROLLARY 4.** *Let  $a$  be a real number. Then, the following hold.*

$$\begin{aligned} \text{(i)} \quad S_q(\sin_q ax)(u) &= \frac{au}{1 + a^2u^2}, \quad |au| < 1; & \text{(ii)} \quad S_q(\cos_q ax)(u) &= \frac{1}{1 + a^2u^2}, \quad |au| < 1, \\ \text{(iii)} \quad L_q(\sin_q ax)(v) &= \frac{a}{v^2 + a^2}, \quad |a| < v; & \text{(iv)} \quad L_q(\cos_q ax)(v) &= \frac{v}{v^2 + a^2}, \quad |a| < v. \end{aligned}$$

From Theorem 1 we state and prove the following corollary.

**COROLLARY 5.** *Let  $a$  be a real number and  $f(x) = \sum_{n \geq 0} A_n x^n$ . Then, we have*

$$N_q\left(x^{\alpha-1} J_{2\mu}^{(1)}(2\sqrt{ax}; q)\right)(u; v) = \frac{(1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} \Gamma_q(\alpha) \sum_{n \geq 0} A_n \frac{u^n}{v^n} (q^\alpha; q)_n.$$

**PROOF** Let  $a$  be a real number, then by aid of (15), we consider to write

$$J_{2\mu}^{(1)}(2\sqrt{ax}; q) = x^\mu \sum_{n \geq 0} \frac{(-1)^n a^{\mu+n}}{(q; q)_{2\mu+n} (q; q)_n} x^n.$$

By replacing  $\alpha$  by  $\alpha - \mu - 1$ , and setting  $A_n = \frac{(-1)^n a^{\mu+n}}{(q; q)_{2\mu+n} (q; q)_n}$ , we, partially, get

$$x^{\alpha-1} J_{2\mu}^{(1)}(2\sqrt{ax}; q) = x^{\alpha-1} \sum_{n \geq 0} A_n x^n = x^{\alpha-1} f(x).$$

Hence, by Theorem 1, we obtain

$$\begin{aligned} N_q\left(x^{\alpha-1} J_{2\mu}^{(1)}(2\sqrt{ax}; q)\right)(u; v) &= \frac{(1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} \sum_{n \geq 0} A_n \frac{u^n}{v^n} (q^\alpha; q)_n \Gamma_q(\alpha) \\ &= \frac{(1-q)^{\alpha-1} u^{\alpha-1}}{v^\alpha} \Gamma_q(\alpha) \sum_{n \geq 0} A_n \frac{u^n}{v^n} (q^\alpha; q)_n. \end{aligned}$$



This completes the proof of the corollary.

In view of Corollary 5, we have the following easy statement.

**COROLLARY 6.** *Let  $a$  be a real number. Then, we have*

$$N_q \left( J_{2\mu}^{(1)} (2\sqrt{ax}; q) \right) (u; v) = \sum_{n \geq 0} A_n u^n (q^\alpha; q)_n.$$

Further, Corollary 5 is expressed in terms of  $q$ -Sumudu and  $q$ -Laplace transforms as in the following results.

**COROLLARY 7.** *Let  $a$  be a positive real number. Then, the following hold.*

$$\begin{aligned} \text{(i)} \quad S_q \left( x^{\alpha-1} J_{2\mu}^{(1)} (2\sqrt{ax}; q) \right) (u) &= (1-q)^{\alpha-1} u^{\alpha-1} \Gamma_q(\alpha) \sum_{n \geq 0} A_n u^n (q^\alpha; q)_n, \\ \text{(ii)} \quad L_q \left( x^{\alpha-1} J_{2\mu}^{(1)} (2\sqrt{ax}; q) \right) (v) &= \frac{(1-q)^{\alpha-1}}{v^\alpha} \Gamma_q(\alpha) \sum_{n \geq 0} A_n \frac{1}{v^n} (q^\alpha; q)_n, \end{aligned}$$

$$\text{where } A_n = (-1)^n \frac{a^{\mu+n}}{(q; q)_{2\mu+n} (q; q)_n}.$$

Corollary 6 extends the results to the case of  $q$ -Sumudu and  $q$ -Laplace transforms as follows.

**COROLLARY 8.** *Let  $a$  be a real number. Then, the following hold.*

$$\text{(i)} \quad S_q \left( J_{2\mu}^{(1)} (2\sqrt{ax}; q) \right) (u) = \sum_{n \geq 0} A_n u^n (q^\alpha; q)_n; \quad \text{(ii)} \quad L_q \left( J_{2\mu}^{(1)} (2\sqrt{ax}; q) \right) (v) = \frac{1}{v} \sum_{n \geq 0} A_n \frac{(q^\alpha; q)_n}{v^n}.$$

## 5 ${}_qN$ Transform of Special Functions

In this section of this paper, we are concerned with the study of the  $q$ -analogue  ${}_qN$  of some special functions. We are precisely concerned with the series representation of the transform and getting some results related to  $q$ -Laplace and  $q$ -Sumudu transforms.

**THEOREM 9.** *Let  $f(x) = \sum_{n \geq 0} A_n x^n$  and  $\alpha > 0$ . Then, we have*

$${}_qN \left( x^{\alpha-1} f(x) \right) (u; v) = \left( \frac{u}{v} \right)^\alpha (1-q)^{\alpha-1} \Gamma_q(\alpha) \sum_{n \geq 0} A_n \frac{(q^\alpha; q)_n}{k \left( \frac{u}{v}; \alpha + n \right)} \left( \frac{u}{v} \right)^n. \quad (37)$$

**PROOF** Let the hypothesis of the theorem be satisfied for some  $\alpha > 0$ . Then, on account of (22), we declare that

$$\begin{aligned} {}_qN \left( x^{\alpha-1} f(x) \right) (u; v) &= \frac{1}{\left( -\frac{v}{u}; q \right)_\infty} \sum_{k \in \mathbb{Z}} q^{\alpha k} \sum_{n \geq 0} A_n q^{kn} \left( -\frac{v}{u}; q \right)_k \\ &= \frac{1}{\left( -\frac{v}{u}; q \right)_\infty} \sum_{n \geq 0} A_n \left( \frac{u}{v} \right)^{\alpha+n} \sum_{k \in \mathbb{Z}} \left( \frac{q^k}{\frac{u}{v}} \right)^{\alpha+n} \left( \frac{-v}{u}; q \right)_k. \quad (38) \end{aligned}$$

Hence, by taking into account the fact that

$$\Gamma_q(\alpha) = \frac{K(A; \alpha)}{(1-q)^{\alpha-1} (-1/A; q)_\infty} \sum_{k \in \mathbb{Z}} \left( \frac{q^k}{A} \right)^\alpha \left( \frac{-1}{A}; q \right)_k$$

where  $K(A; \alpha) = A^{\alpha-1} \frac{(-q/\alpha; q)_\infty}{(-q^t/\alpha; q)_\infty} \frac{(-\alpha; q)_\infty}{(-\alpha q^{1-t}; q)_\infty}$  [25, Equ.(24)] (for  $A = \frac{u}{v}$  and  $\alpha =$

$\alpha + n$ ), we get

$$\begin{aligned} {}_qN(x^{\alpha-1}f(x))(u;v) &= \sum_{k \in \mathbb{Z}} A_n \left(\frac{u}{v}\right)^{\alpha+n} \frac{\Gamma_q(\alpha+n)(1-q)^{\alpha+n-1}}{K\left(\frac{u}{v}; \alpha+n\right)} \\ &= \left(\frac{u}{v}\right)^{\alpha} (1-q)^{\alpha-1} \Gamma_q(\alpha) \sum_{n \geq 0} A_n \frac{(q^{\alpha}; q)_n}{K\left(\frac{u}{v}; \alpha+n\right)} \left(\frac{u}{v}\right)^n. \end{aligned}$$

This completes the proof of the theorem.

As a straightforward corollary of Theorem 9, the previous theorem (for  $A_0 = 1, A_n = 0$ , for  $n \geq 1$ ) gives

$${}_qN(x^{\alpha-1})(u;v) = \frac{\left(\frac{u}{v}\right)^{\alpha-1} (1-q)^{\alpha-1} \Gamma_q(\alpha)}{K\left(\frac{u}{v}; \alpha\right)}. \quad (39)$$

Also, Equ. (39) reveals :

COROLLARY 10. *The following hold true.*

$$(i) \quad {}_qL(x^{\alpha-1})(v) = \frac{1}{v^{\alpha-1}} \frac{(1-q)^{\alpha-1} \Gamma_q(\alpha)}{K\left(\frac{1}{v}; \alpha\right)}. \quad (40)$$

$$(ii) \quad {}_qS(x^{\alpha-1})(u) = \frac{u^{\alpha-1} (1-q)^{\alpha-1} \Gamma_q(\alpha)}{K(u; \alpha)}. \quad (41)$$

Further, (39) and (41) jointly lead to the conclusion  $({}_qN(1))(u;v) = \frac{1}{K\left(\frac{u}{v}; \alpha\right)}$ . Therefore, we are directed to the results

$${}_qL(1)(v) = \frac{1}{K(v;1)} \text{ and } ({}_qS(1))(u) = \frac{1}{K(u;1)}.$$

THEOREM 11. *Let  $a$  be a real number and  $f(x) = {}_{m-k}\Phi_{m-1} \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k}, q^{\alpha} \\ b_1, b_2, \dots, b_{m-1} \end{matrix} \middle| q, ax \right]$ . Then, we have*

$$\begin{aligned} {}_qN(x^{\alpha-1}f(x))(u;v) &= \frac{\left(\frac{u}{v}\right)^{\alpha} (1-q)^{\alpha-1} \Gamma_q(\alpha)}{K\left(\frac{u}{v}; \alpha\right)} \\ &\quad {}_{m-k+1}\Phi_{m-1} \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k}, q^{\alpha} \\ b_1, b_2, \dots, b_{m-1} \end{matrix} \middle| q, \frac{au}{vq^{\alpha}} \right]. \end{aligned}$$

PROOF Let the hypothesis of the theorem be satisfied. A charity of (14) gives

$$f(x) = \sum_0^{\infty} \frac{(a_1, a_2, \dots, a_{m-k}; q)_n}{(b_1, b_2, \dots, b_{m-1}, q)_n} \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^k \frac{a^n}{(q; q)_n} x^n.$$

On setting  $A_n = \frac{(a_1, a_2, \dots, a_{m-k}; q)_n}{(b_1, b_2, \dots, b_{m-1}, q)_n} \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^k \frac{a^n}{(q; q)_n}$  and using Theorem 9 it implies

$${}_qN(x^{\alpha-1}f(x))(u;v) = \left(\frac{u}{v}\right)^{\alpha} (1-q)^{\alpha-1} \Gamma_q(\alpha) \sum_0^{\infty} A_n \frac{(q^{\alpha}; q)_n}{K\left(\frac{u}{v}; \alpha+n\right)} \left(\frac{u}{v}\right)^n.$$

By using the fact that  $K(A, \alpha) = q^{\alpha-1} K(A, \alpha-1)$ , the preceding equation gives

$$\begin{aligned} {}_qN(x^{\alpha-1}f(x))(u;v) &= \left(\frac{u}{v}\right)^\alpha \frac{(1-q)^{\alpha-1} \Gamma_q(\alpha)}{K\left(\frac{u}{v}; \alpha\right)} \sum_0^\infty A_n(q^\alpha; q)_n \\ &\quad \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{-1} \left(\frac{-1}{q^\alpha}\right)^n \left(\frac{u}{v}\right)^n \\ &= \frac{\left(\frac{u}{v}\right)^\alpha (1-q)^{\alpha-1} \Gamma_q(\alpha)}{K\left(\frac{u}{v}; \alpha\right)} \\ &\quad {}_{m-k+1}\Phi_{m-1} \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k}, q^\alpha \\ b_1, b_2, \dots, b_{m-1} \end{matrix} \middle| q, \frac{au}{vq^\alpha} \right]. \end{aligned}$$

Hence the theorem is completely proved.

**COROLLARY 12.** *Let  $a$  be a real number and  $f(x)$  be defined in terms of the  $q$ -hypergeometric function*

$$f(x) = {}_{m-k+1}\Phi_{m-1} \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k} \\ b_1, b_2, \dots, b_{m-1} \end{matrix} \middle| q; ax \right].$$

*Then, the following identities hold.*

$$\begin{aligned} \text{(i)}_q L(x^{\alpha-1}f(x))(v) &= \frac{(1-q)^{\alpha-1} \Gamma_q(\alpha)}{K\left(\frac{1}{v}; \alpha\right)} {}_{m-k+1}\Phi_{m-1} \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k}, q^\alpha \\ b_1, b_2, \dots, b_{m-1} \end{matrix} \middle| q, \frac{a}{vq^\alpha} \right]. \\ \text{(ii)}_q S(x^{\alpha-1}f(x))(u) &= \frac{u^\alpha (1-q)^{\alpha-1} \Gamma_q(\alpha)}{K(u; \alpha)} {}_{m-k+1}\Phi_{m-1} \left[ \begin{matrix} a_1, a_2, \dots, a_{m-k}, q^\alpha \\ b_1, b_2, \dots, b_{m-1} \end{matrix} \middle| q, \frac{au}{q^\alpha} \right]. \end{aligned}$$

PROOF is straightforward from Theorem 11. Details are therefore omitted.

Let  $\alpha = m = k = 1$  and  $a > 0$ . Then, Corollary 12 gives

$${}_qN(E_q(ax))(u;v) = \frac{u}{vK\left(\frac{u}{v}, 1\right)} {}_1\Phi_0 \left[ q \middle| q, -\frac{au}{qv} \right].$$

Hence, it follows

$$\begin{aligned} \text{(i)}_q L(E_q(ax))(v) &= \frac{1}{vK\left(\frac{1}{v}, 1\right)} {}_1\Phi_0 \left[ q \middle| q, -\frac{a}{qv} \right]. \\ \text{(ii)}_q S(E_q(ax))(u) &= \frac{u}{K(u, 1)} {}_1\Phi_0 \left[ q \middle| q, -\frac{au}{q} \right]. \end{aligned}$$

Also, readers may easily verify that

$${}_1\Phi_0 \left[ q \middle| q, -\frac{au}{qv} \right] = \sum_0^\infty \left(\frac{au}{qv}\right)^n = \frac{au}{qv+au}, \quad |au| < qv. \quad (42)$$

From above and the fact that  $K(s, 1) = 1$  we have the following corollary.

**COROLLARY 13.** *Let  $a$  be a real number. Then, we have*

$${}_qN(E_q(ax))(u, v) = \frac{qu}{(qv+au)}, \quad |au| < qv. \quad (43)$$

Hence, (43) indeed reveals

$$\text{(i)}_q L(E_q(ax))(v) = \frac{q}{(qv+a)}, \quad |a| < qv; \quad \text{(ii)}_q S(E_q(ax))(u) = \frac{qu}{(q+au)}, \quad |au| < q.$$

It may also be mentioned here that Corollary 13 and the identities

$${}_q \sin x = \frac{E_q(ix) - E_q(-ix)}{2i} \quad \text{and} \quad {}_q \cos x = \frac{E_q(ix) + E_q(-ix)}{2} \quad (44)$$

state, without proof, the following result.

COROLLARY 14. *Let  $a$  be a real number. Then, we have*

$$(i) {}_qN({}_q \sin ax)(u; v) = \frac{-qau^2}{q^2v^2 + a^2u^2}, |au| < qv.$$

$$(ii) {}_qN({}_q \cos ax)(u; v) = \frac{q^2uv}{q^2v^2 + a^2u^2}, |au| < qv.$$

Further, Corollary 14 suggests to have the following conclusions proclaimed.

$$(i) {}_qL({}_q \sin ax)(v) = \frac{-qa}{q^2v^2 + a^2}, |a| < qv; \quad (ii) {}_qL({}_q \cos ax)(v) = \frac{q^2v}{q^2v^2 + a^2},$$

$$|a| < qv.$$

$$(iii) {}_qS({}_q \sin ax)(u) = -\frac{qa}{q^2 + a^2u^2}, |au| < qv; \quad (iv) {}_qS({}_q \cos ax)(u) = \frac{q^2}{q^2 + a^2u^2},$$

$$|au| < qv.$$

Further results concerning some other special functions can be obtained similarly.

## 6 $q$ -Analogues of $N$ -Transforms for the $q$ -Fox's $H$ -Function

Let  $\alpha_j$  and  $\beta_j$  be positive integers and  $0 \leq m \leq N$ ;  $0 \leq n \leq M$ . Due to [27], the  $q$ -analogue of the Fox's  $H$ -function is given as

$$H_{M,N}^{m,n} \left[ x; q \left| \begin{matrix} (a_1, \alpha_1), (a_2, \alpha_2), \dots, (a_m, \alpha_m) \\ (b_1, \beta_1), (b_2, \beta_2), \dots, (b_N, \beta_N) \end{matrix} \right. \right] =$$

$$\frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^m G(q^{b_j - \beta_j s}) \prod_{j=1}^n G(q^{1 - a_j + \alpha_j s}) \pi x^s}{\prod_{j=m+1}^N G(q^{1 - b_j + \beta_j s}) \prod_{j=n+1}^M G(q^{a_j - \alpha_j s}) G(q^{1-s}) \sin \pi s} d_q s \quad (45)$$

where  $G$  is defined in terms of the product

$$G(q^\alpha) = \prod_{k=0}^{\infty} (1 - q^{\alpha - k})^{-1} = \frac{1}{(q^\alpha, q)_\infty}. \quad (46)$$

The contour  $C$  is parallel to  $\operatorname{Re}(ws) = 0$ , with indentations in such away all poles of  $G(q^{b_j - \beta_j s})$ ,  $1 \leq j \leq m$ , are its right and those of  $G(q^{1 - a_j + \alpha_j s})$ ,  $1 \leq j \leq n$ , are the left of  $C$ . The integral converges if  $\operatorname{Re}(s \log x - \log \sin \pi s) < 0$ , for large values of  $|s|$  on  $C$ . Hence,

$$|\arg(x) - w_2 w_1^{-1} \log |x|| < \pi, \quad |q| < 1, \quad \log q = -w = -w_1 - iw_2,$$

where  $w_1$  and  $w_2$  are real numbers.

Indeed, for  $\alpha_i = \beta_j = 1$ , for all  $i, j$ , (45) gives the  $q$ -analogue of the Meijer's  $G$ -function

$$G_{M,N}^{m,n} \left[ x; q \left| \begin{matrix} a_1, a_1, \dots, a_M \\ b_1, b_2, \dots, b_N \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^m G(q^{b_j - s}) \prod_{j=1}^n G(q^{1 - a_j + s}) \pi x^2}{\prod_{j=m+1}^N G(q^{1 - b_j + s}) \prod_{j=n+1}^M G(q^{a_j - s}) G(q^{1-s}) \sin \pi s} d_q s$$

where  $0 \leq m \leq N$ ;  $0 \leq n \leq M$  and  $\operatorname{Re}(s \log x - \log \sin \pi s) < 0$ .

We have the following main result of this section.

THEOREM 15. (i) Let  $\lambda$  be any complex number and  $k \in (0, \infty)$ . The  $q$ -Natural transform  $N_q$  of the Fox's  $H_q$ -Function is given as

$$N_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), b_2, \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (u; v) = \frac{u^\lambda}{v^{\lambda+1} G(q)} H_{M+1,N}^{m,n+1} \left[ \lambda \frac{u^k}{v^k}; q \left| \begin{matrix} (-\lambda, k), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right].$$

(ii) Let  $\lambda$  be any complex number and  $k \in (-\infty, 0)$ . The  $q$ -Natural transform  $N_q$  of the Fox's  $H_q$ -Function is given as

$$N_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), b_2, \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (u; v) = \frac{u^\lambda}{v^{\lambda+1} G(q)} H_{M,N+1}^{m+1,n} \left[ \lambda \frac{u^k}{v^k}; q \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (1 + \lambda, -k), (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right].$$

PROOF We prove Part (i) since proof of Part (ii) is similar. By considering (45), we have

$$N_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), b_2, \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (u; v) = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^m G(q^{b_j - \beta_j z}) \prod_{j=1}^n G(q^{1 - a_j + \alpha_j z}) \pi(\lambda)^z}{\prod_{j=m+1}^N G(q^{1 - b_j + \beta_j z}) \prod_{j=n+1}^M G(q^{a_j - \alpha_j z}) G(q^{1-z}) \sin \pi z} N_q(x^{\lambda+kz})(u; v) d_q z. \quad (49)$$

By virtue of Theorem 2, (49) gives

$$N_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), b_2, \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (u; v) = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^m G(q^{b_j - \beta_j z}) \prod_{j=1}^n G(q^{1 - a_j + \alpha_j z}) \pi(\lambda)^z}{\prod_{j=m+1}^N G(q^{1 - b_j + \beta_j z}) \prod_{j=n+1}^M G(q^{a_j - \alpha_j z}) G(q^{1-z}) \sin \pi z} \frac{(1-q)^{\lambda+kz}}{v^{\lambda+kz+1}} \Gamma_q(\lambda + kz + 1) d_q z.$$

The fact that  $(1-q)^{\lambda+kz} \Gamma_q(\lambda + kz + 1) = \frac{G(q)^{\lambda+kz}}{G(q)}$  gives

$$\begin{aligned} N_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), b_2, \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (u; v) &= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^m G(q^{b_j - \beta_j z}) \prod_{j=1}^n G(q^{1 - a_j + \alpha_j z}) \prod(\lambda)^z u^{\lambda+kz}}{\prod_{j=m+1}^N G(q^{1 - b_j + \beta_j z}) \prod_{j=n+1}^M G(q^{a_j - \alpha_j z}) G(q^{1-z}) \sin \pi z v^{\lambda+kz+1}} \frac{G(q^{\lambda+kz})}{G(q)} d_q z \\ &= \frac{1}{2\pi i} \frac{u^\lambda}{v^{\lambda+1} G(q)} \int_C \frac{\prod_{j=1}^m G(q^{b_j - \beta_j z}) \prod_{j=1}^n G(q^{1 - a_j + \alpha_j z}) G(q^{\lambda+kz}) \prod \left( \lambda \frac{u^k}{v^k} \right)^z}{\prod_{j=m+1}^N G(q^{1 - b_j + \beta_j z}) \prod_{j=n+1}^M G(q^{a_j - \alpha_j z}) G(q^{1-z}) \sin \pi z} d_q z \\ &= \frac{u^\lambda}{v^{\lambda+1} G(q)} H_{M+1,N}^{m,n+1} \left[ \lambda \frac{u^k}{v^k}; q \left| \begin{matrix} (-\lambda_1, k), (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right], \end{aligned}$$

for  $k > 0$

This completes the proof of Part (i) of the theorem. Proof of Part (ii) is quite similar.

Hence the theorem is completely proved.

COROLLARY 16. (a) Let  $\lambda$  be any complex number and  $k \in (0, \infty)$ . Then, we have

$$\begin{aligned}
& \text{(i)} L_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (v) = \frac{1}{v^{\lambda+1} G(q)} \times \\
& \quad H_{M+1,N}^{m,n+1} \left[ \frac{\lambda}{v^k}; q \left| \begin{matrix} (-\lambda_1, k), (a_1, \alpha_1), \dots, (a_\mu, \alpha_\mu) \\ (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right]. \\
& \text{(ii)} S_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (u) = \frac{u^\lambda}{G(q)} \times \\
& \quad H_{M+1,N}^{m,n+1} \left[ \lambda, u^k; q \left| \begin{matrix} (-\lambda_1, k), (a_1, \alpha_1), \dots, (a_\mu, \alpha_\mu) \\ (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right]. \\
& \text{(b)} \text{ Let } \lambda \text{ be any complex number and } k \in (-\infty, 0). \text{ Then, we have} \\
& \text{(i)} L_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (v) = \frac{1}{v^{\lambda+1} G(q)} \times \\
& \quad H_{M,N+1}^{m+1,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (1+\lambda, -k), (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right]. \\
& \text{(ii)} S_q \left( x^\lambda H_{M,N}^{m,n} \left[ \lambda x^k; q \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right] \right) (u) = \frac{u^\lambda}{G(q)} \times \\
& \quad H_{M,N+1}^{m+1,n} \left[ \lambda u^k; q \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_M, \alpha_M) \\ (1+\lambda, -k), (b_1, \beta_1), \dots, (b_N, \beta_N) \end{matrix} \right. \right].
\end{aligned}$$

## 7 Concrete Examples

By virtue of Theorem 15 and the elementary extentions of some  $q$ -analogues of  $\sin_q x$ ,  $\cos_q x$ ,  $\sinh_q x$  and  $\cosh_q x$  in terms of Fox's H-function; see [26], we introduce the following examples.

EXAMPLE 1. Let  $k = 2$  and  $\lambda = \frac{(1-q)^2}{4}$  in Theorem 15, then we have

$$\begin{aligned}
N_q \left( x^{\frac{(1-q)^2}{4}} \sin_q x \right) (u; v) &= \frac{\sqrt{\pi} (1-q)^{-\frac{1}{2}} G(q) u^{(1-q)^2/4}}{v^{\frac{(1-q)^2}{4}+1}} \\
& \quad H_{1,3}^{1,1} \left[ \frac{(1-q)^2}{4} \frac{u^2}{v^2}; q \left| \begin{matrix} -\frac{(1-q)^2}{4}, 2 \\ (\frac{1}{2}, 1), (0, 1), (1, 1) \end{matrix} \right. \right].
\end{aligned}$$

EXAMPLE 2. On setting  $k = 2$  and  $\lambda = \frac{(1-q)^2}{4}$  in Theorem 15, we get

$$\begin{aligned}
N_q \left( x^{\frac{(1-q)^2}{4}} \cos_q x \right) (u; v) &= \frac{\sqrt{\pi} (1-q)^{-\frac{1}{2}} G(q) u^{(1-q)^2/4}}{v^{\frac{(1-q)^2}{4}+1}} \\
& \quad H_{1,3}^{1,1} \left[ \frac{(1-q)^2}{4} \frac{u^2}{v^2}; q \left| \begin{matrix} -\frac{(1-q)^2}{4}, 2 \\ (0, 1), (\frac{1}{2}, 1), (1, 1) \end{matrix} \right. \right].
\end{aligned}$$

EXAMPLE 3. On setting  $k = 2$  and  $\lambda = \frac{(1-q)^2}{4}$ , in Theorem 15 we get

$$\begin{aligned}
N_q \left( x^{\frac{(1-q)^2}{4}} \sinh_q x \right) (u; v) &= \frac{\sqrt{\pi} (1-q)^{-\frac{1}{2}} G(q) u^2}{iv^2} \\
& \quad H_{1,3}^{1,1} \left[ \frac{-(1-q)^2 u^2}{4v^2}; q \left| \begin{matrix} \left( \frac{(1-q)^2}{4}, 2 \right) \\ (\frac{1}{2}, 1), (0, 1), (1, 1) \end{matrix} \right. \right].
\end{aligned}$$

EXAMPLE 4. On setting  $k = 2$ ,  $\lambda = \frac{(1-q)^2}{4}$ , Theorem 15 gives

$$N_q \left( x^{\frac{(1-q)^2}{4}} \cosh_q x \right) (u; v) = \frac{\sqrt{\pi} (1-q)^{\frac{-1}{2}} G(q) u^2}{v^2} H_{1,3}^{1,1} \left[ \frac{-(1-q)^2 u^2}{4v^2}; q \left| \begin{matrix} \left( \frac{(1-q)^2}{4}, 2 \right) \\ (0, 1), \left( \frac{1}{2}, 1 \right), (1, 1) \end{matrix} \right. \right].$$

For similar results of  $S_q$  and  $L_q$ , we set  $u = 1$  and  $v = 1$  in the preceeding Examples.

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